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A QUASILINEARIZATION APPROACH TO THE SOLUTION OF ELASTIC BEAMS ON NONLINEAR FOUNDATIONS

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Abstract—In this paper, the solution of a beam on nonlinear elastic foundation whose deflection satisfies the nonlinear boundary value problem (1, 2), is studied by means of the theory of quasilinearization. The problem is formulated in Section 2 where conditions for the existence and uniqueness of the solution are stated. In Section 3, the idea of quasilinearization is introduced and the positivity of an associated linear differential operator is investigated. In Section 4 the usual version of quasilinearization, i.e. the Newton–Raphson–Kantorovich sequence, is presented and conditions under which this sequence is monotonically convergent, are established. In Section 5, an alternative successive approximation scheme whose derivation relies on ideas of quasilinearization, is presented. Finally, an example is solved by numerical procedures based in the methods discussed in previous sections.

1. INTRODUCTION

In a recent paper, E. S. Lee, C. L. Huang and I. K. Hwang [1], discussed the use of quasilinearization for the numerical solution of a class of problems in elastic bar theory. In this paper we wish to extend those ideas by presenting additional properties and methods based on quasilinear approaches in the context of a related problem in structural mechanics. The equation chosen to present these ideas is that of a beam on nonlinear foundation given by equations (1 and 2). The rationale behind this choice lies mainly in the fact that fourth order equations such as (1 and 2) seems not to have been extensively investigated from the viewpoint of this paper. In addition, we wish to offer an appropriate illustrative example of some ideas in quasilinearization using a modicum of analytical effort, a condition afforded by the present equation.

The equation is presented in Section 2 where conditions for existence and uniqueness of the solution are stated. In Section 3 we introduce the idea of quasilinearization and discuss in some detail the monotonic behavior of a certain associated linear operator. In Section 4, the usual version of quasilinearization, the Newton–Raphson–Kantorovich sequence, is presented. Conditions under which this sequence is monotonically convergent, are established. In Section 5, motivated by some of the typical computational difficulties in the application of the Newton–Raphson–Kantorovich method, we introduce an alternative, successive approximation scheme whose derivation again relies on ideas of quasilinearization. Finally, in Section 6, an example is presented and thoroughly investigated by numerical methods.

2. THE NONLINEAR PROBLEM

We consider the equation of a beam on nonlinear foundation

$$\frac{d^4u}{dx^4} + g(u, x) = p(x), \tag{1}$$

where u is the deflection, $p(x)$ the external load acting on the beam and $g(u, x)$ a nonlinear function of u representing the reaction of the nonlinear foundation. In order to illustrate the ideas we shall assume that the ends of the beam are hinged. Therefore, u must satisfy the two point value condition

$$\begin{aligned} u(0) &= u''(0) = 0, \\ u(L) &= u(''L) = 0, \end{aligned} \tag{2}$$

where L is the length of the beam. Boundary conditions other than (2) can be treated equally well with the methods presented below.

We further assume that $g(u, x)$ is continuous in u and possesses a bounded first derivative satisfying the condition

$$\left|\frac{dg}{du}\right| < \frac{\pi^4}{L^4}, \tag{3}$$

where $\frac{\pi^4}{L^4}$ stands as the first characteristic value of the operator

$$\frac{d^4u}{dx^4} - \lambda u = 0, \tag{4}$$

associated with boundary conditions (2). Under these conditions u satisfying (1 and 2) exists and is unique. This can be proved directly by using fixed points techniques or by transforming (1 and 2) into a nonlinear integral equation and then resorting to standard results for the resulting Hammerstein equation[2].

3. QUASILINEARIZATION. MONOTONE BEHAVIOR

Assuming that g is convex in u we can write

$$g(u, x) = \max_v [g(v, x) + (u - v)g_v(v, x)], \tag{5}$$

where g_v stands for $\frac{\partial g}{\partial v}$ and where the minimization is carried over all functions v defined in the interval $[0, L]$. The maximum in (5) is attained at $v = u$. Substituting (5) in (1) we obtain

$$\frac{d^4u}{dx^4} + g(v, x) + (u - v)g_v(v, x) \leq p(x). \tag{6}$$

We now compare u satisfying (1, 2 and 6), with w satisfying the equation

$$\frac{d^4w}{dx^4} + g(v, x) + (w - v)g_v(v, x) = p(x), \tag{7}$$

and boundary conditions similar to (2). To this end we form the difference

$$z = w - u, \tag{8}$$

which clearly satisfies the functional inequality

$$\frac{d^4 z}{dx^4} + g_v z \geq 0, \tag{9}$$

and the homogeneous boundary conditions

$$\begin{aligned} z(0) = z''(0) = 0, \\ z(L) = z''(L) = 0. \end{aligned} \tag{10}$$

Now, if from the differential inequality (9 and 10) we can infer that $z \geq 0$, then it is clear that any choice of v in (7) automatically furnishes an upper bound on u. It is therefore of interest to determine conditions under which this positivity condition holds.

Instead of (9) we consider the equality equation

$$\frac{d^4 z}{dx^4} + g_v z = f, \tag{11}$$

where f is a nonnegative function of x. Now, if we can find two positive numbers λ_1 and λ_2 such that

$$\lambda_1 \geq g_v \geq -\lambda_2, \tag{12}$$

and such that the operators

$$L(y_1) = \frac{d^4 y_1}{dx^4} + \lambda_1 y_1 = f, \quad y_1(0) = y_1''(0) = y_1(L) = y_1''(L) = 0, \tag{13}$$

and

$$L(y_2) = \frac{d^4 y_2}{dx^4} - \lambda_2 y_2 = f, \quad y_2(0) = y_2''(0) = y_2(L) = y_2''(L) = 0, \tag{14}$$

are nonnegative in the sense that, for all $f \geq 0$, $L(y_1) = f \rightarrow y_1 \geq 0$, and $L(y_2) = f \rightarrow y_2 \geq 0$, then z in (10 and 11), is nonnegative. To prove this we make

$$\lambda_1 - h = g_v, \tag{15}$$

where, by virtue of equation (12), h is nonnegative. Substituting g_v given by (15) in (11) we obtain

$$\frac{d^4 z}{dx^4} + \lambda_1 z = hz + f. \tag{16}$$

Therefore

$$z(x) = \int_0^L G(x, y) h(y) z(y)\, dy + \alpha(x), \tag{17}$$

where $G(x, y)$, the Green's function associated with the operator (13), is a pointwise nonnegative kernel in the region $0 \leq x, y \leq L$, because of the assumed nonnegativity of the differential operator (13). Hence, $\alpha(x)$ in (17), given by

$$\alpha(x) = \int_0^L G(x, y) f(y)\, dy, \tag{18}$$

is nonnegative. Now, the Neumann's series

$$z(x) = \alpha(x) + \int_0^L G(x, y)\, h(y)\, \alpha(y)\, \mathrm{d}y + \int_0^L \int_0^L G(x, y)\, G(y, \xi)\, h(y)\, h(\xi)\, \alpha(\xi)\, \mathrm{d}y\, \mathrm{d}\xi + \cdots \tag{19}$$

obtained by iteration in (17), converges provided h satisfies the condition

$$\sup_x h < \lambda_0 \tag{20}$$

where λ_0 is the smallest characteristic value of the operator

$$z(x) - \lambda \int_0^L G(x, y) z(y)\, \mathrm{d}y = 0, \tag{21}$$

a condition that it is certainly verified if

$$\inf_x g_v > -\frac{\pi^4}{L^4}. \tag{22}$$

The required nonnegativity of z follows immediately from the nonnegativity of each term of the Neumann's series (19).

It remains now to provide estimates for λ_1 and λ_2 in equation (12). It is easy to show that a lower bound on λ_2 is

$$-\lambda_2 > -\left(\frac{\pi}{L}\right)^4, \tag{23}$$

i.e., if λ_2 satisfies equation (23), then y_2 in equation (14) is nonnegative. This result follows from the representation of y_2 in terms of the Neumann's series

$$y_2(x) = f(x) + \lambda_2 \int_0^L K(x, y) f(y)\, \mathrm{d}y + \lambda_2^2 \int_0^L \int_0^L K(x, y) K(y, \xi) f(\xi)\, \mathrm{d}y\, \mathrm{d}\xi + \cdots, \tag{24}$$

where $K(x, y)$ is the Green's function associated with the operator

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = f, \quad y(0) = y''(0) = y(L) = y''(L) = 0, \tag{25}$$

which is clearly positive. The lower bound for $-\lambda_2$ given by (23) is best possible, since for $\lambda_2 > \left(\frac{\pi}{L}\right)^4$, the solution of (14), with $f = \sin \frac{\pi}{L} x$, is negative.

A conservative estimate on the upper bound λ_1 of g_v may be shown to be $\left(\frac{\pi}{L}\right)^4$. This, of course, is not best possible† but it is certainly enough for our purposes. In fact, in com-

† The greatest upper bound on λ_1 may be shown to be $4\lambda^4$ where λ is the smallest positive root of $tg\lambda = th\lambda$. See [6 and 7] where in addition to this, the reader will find an approach to study the positive solutions of general linear fourth order equations.

paring (3, 22, 23 and 26) we conclude that the conditions for nonnegativity of the fundamental operator (9 and 10) hold with the maximum interval of validity, i.e. for

$$|g_v| < \left(\frac{\pi}{L}\right)^4. \tag{27}$$

Clearly, if $g(u, x)$, instead of being a convex function in u, is a concave one, then, if g_v satisfies (27), any choice of v in (7) will furnish a lower bound w on u.

4. SUCCESSIVE APPROXIMATIONS. I

The procedure to determine upper bounds exposed in the preceding section may be combined with the method of successive approximations in various fashions. The usual one consists in the construction of a sequence of functions $u_n(x)$, $n = 1, 2, \ldots$, by using the iterative scheme

$$\frac{d^4u_{n+1}}{dx^4} + g(u_n, x) + (u_{n+1} - u_n)g_{u_n}(u_n, x) = p(x), \tag{28}$$

where the first element of the sequence $u_0(x)$, is a given function. Clearly, equation (28) has been derived from (7) by making $w = u_{n+1}$ and $v = u_n$, i.e. by using the last approximation to generate the new one. When $g(u, x)$ satisfies the convexity (concavity) condition

$$g_{uu} > 0, \qquad (g_{uu} < 0), \tag{29}$$

and the additional condition

$$|g_u| < \frac{\pi^4}{L^4}, \tag{30}$$

that insures the positivity of the associated linear operator (9 and 10), the sequence u_n, $n = 1, 2, \ldots$, generated by (28) and pertinent boundary conditions, is monotonically convergent from above (below) to the function u satisfying the original boundary value problem (1 and 2). In addition the convergence is quadratic, i.e.

$$\sup_x |u_{n+1} - u| \leq k \sup_x |u_n - u|^2, \tag{31}$$

where k is a constant independent of n. The proof of these properties follow standard patterns[3] and will not be repeated here.

We observe that equation (28) could have been directly derived by considering a Taylor's series expansion of $g(u, x)$ up to linear terms in u, and without resorting to any convexity or positive property of the operator equation. If, in fact, g does not satisfy (29 to 30), the sequence u_n generated by (28) will still be quadratically convergent, if convergent at all. This is in effect, the basis of the Newton–Raphson–Kantorovich method to solve functional equations. In this method, convergency will be strongly dependent on the initial approximation chosen to start the process and, of course, unless additional conditions are stated, we cannot insure monotonicity of the approximations.

5. SUCCESSIVE APPROXIMATIONS. II

One of the difficulties associated with the use of the Newton–Raphson–Kantorovich scheme such as (28) is that in order to compute the element u_{n+1} of the sequence we need the previous function u_n available in fast storage. This is certainly not a problem in the present case but it is a severe limitation when we deal with higher dimensional problems. To remove part of the difficulties associated with storage, we may regard (28) and (the associated boundary conditions) not as a single fourth order linear differential equation in u_{n+1}, where u_n is a given function, but as a system of n fourth order linear differential equations in $u_1, u_2, \ldots u_n$, u_{n+1}, subject to boundary conditions. The initial approximation $u_0(x)$ is usually given as a constant or in terms of functions easy to generate. In this method we need not to store u_n while computing u_{n+1}, because not only u_n but also the previous ones $u_1, u_2, \ldots u_{n+1}$, are being simultaneously computed. It is therefore clear that in order to save storage we are increasing the computational time. In fact, to determine the element u_n by using this method, we are required to integrate $\frac{1}{2}n(n+1)$ fourth order linear equations subject to two point conditions, as opposed to n of such equations when we store the previous approximations.

In order to avoid storage problems, and, *at the same time*, keep the number of equations to integrate at a minimum, we can use a modified successive approximations scheme. The starting point for such a method will be the associated linear equation (7), derived from quasilinearization. Using equation (7) we generate the successive approximation process

$$\frac{d^4 u_{n+1}}{dx^4} + g(v_n, x) + (u_{n+1} - v_n) g_{v_n}(v_n, x) = p(x), \tag{32}$$

where u_{n+1} satisfies the boundary conditions of the beam given by

$$\begin{aligned} u_{n+1}(0) &= u''_{n+1}(0) = 0, \\ u_{n+1}(L) &= u''_{n+1}(L) = 0, \end{aligned} \tag{33}$$

and where v_n is the function obtained by integrating the initial value problem

$$\frac{d^4 v_n}{dx^4} + g(v_n, x) = q(x),$$

$$\begin{aligned} v_n(0) &= u_n(0), \\ v'_n(0) &= u'_n(0), \\ v''_n(0) &= u''_n(0), \\ v'''_n(0) &= u'''_n(0). \end{aligned} \tag{34}$$

The solution of the linear two-point value problem can be obtained by reduction to a Cauchy problem by employing the well known technique of superposition of principal solutions, or invariant imbedding or any other method. The resulting initial-value system, together with equation (34) for v_n, is finally integrated by using one of the various standard routines for differential equations furnishing u_{n+1}, v_n and their derivatives. The distance between u_{n+1} and v_n may be used as a stopping rule. Since to start new iterations, say that of order $n+2$, all we need are the values of $u'_{n+1}(0)$ and $u'''_{n+1}(0)$, a very convenient method to integrate (32 and 33) is by invariant imbedding. See for example[4].

The method exposed above was first presented in connection with the solution of non-linear integro–differential systems appearing in problems of design of structures in the presence of creep[5].

6. NUMERICAL EXAMPLE

To illustrate the methods and ideas previously exposed, we consider the nonlinear equation

$$\frac{d^4u}{dx^4} + g(u) = \pi^4 \sin \pi x + g(\sin \pi x), \tag{35}$$

subject to boundary conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0, \tag{36}$$

whose (unique) solution is given by

$$u = \sin \pi x, \tag{37}$$

for any function g satisfying equation (3). For numerical purposes we consider

$$g(u) = u + c\, e^{du}, \tag{38}$$

where c and d are constants such that

$$|g_u| = |1 + cd\, e^{du}| < \pi^4. \tag{39}$$

When $c > 0$ ($c < 0$), $g(u)$ is convex (concave). Therefore, by using the results of Sections 3 and 4, when $c > 0$ the approximating sequence derived by quasilinearization will monotonically converge from above (below) to the function $\sin \pi x$.

The Newton–Raphson–Kantorovich sequence associated with equations (35 and 36), where g is given by (38), is

$$\frac{d^4u_{n+1}}{dx^4} + (1 + cd\, e^{du_n})u_{n+1} = q(x) - c\, e^{du_n}(1 - du_n),$$

$$u_{n+1}(0) = u''_{n+1}(0) = u_{n+1}(1) = u''_{n+1}(1) = 0, \tag{40}$$

where $q(x)$ is given by

$$q(x) = (1 + \pi^4) \sin \pi x + c\, e^{d \sin \pi x}. \tag{41}$$

The solution of equation (40) was numerically obtained for two sets of values of c and d as follows

$$c = 5{\cdot}0, \qquad d = 2{\cdot}0,$$

and (42)

$$c = -0{\cdot}1, \qquad d = 1{\cdot}0.$$

In both cases the initial approximation chosen to start the process was $u_0 \equiv 0$. Equation (40) was solved using the method of superposition of principal solutions numerically obtained by integration of the associated initial value problems using an Adams–Moulton integration scheme and various step sizes for comparison purposes. The results are displayed in Tables 1 and 2, where only the value of the successive approximations at $x = 0{\cdot}5$, i.e. $u_n(0.5)$, is presented. The exact value in both cases is $u(0{\cdot}5) = 1{\cdot}0$. The computations were performed for four step sizes namely $h = 0{\cdot}010$, $0{\cdot}005$, $0{\cdot}002$ and $0{\cdot}001$, using an Adams–Moulton scheme on a CDC 6400 computer. For each step size the number of accurate figures was determined separately using a similar equation whose exact solution was known. Only the figures estimated to be exact are shown in the tables.

Table 1

u_n	Newton–Raphson–Kantorovich $c = 5\cdot0$, $d = 2\cdot0$ $h = 0\cdot010$	$h = 0\cdot005$	$h = 0\cdot002$	$h = 0\cdot001$
$u_0(x)$	0·0 for $x\varepsilon[0, 1]$	0·0 for $x\varepsilon[0, 1]$	0·0 for $x\varepsilon[0, 1]$	0·0 for $x\varepsilon[0, 1]$
$u_1(0\cdot5)$	1·156831	1·1568311	1·156831167	1·1568311671
$u_2(0\cdot5)$	1·008793	1·0087937	1·008793789	1·0087937892
$u_3(0\cdot5)$	1·000025	1·0000255	1·000025558	1·0000255582
$u_4(0\cdot5)$	1·000000	1·0000000	1·000000000	1·0000000002
$u_5(0\cdot5)$	1·000000	1·0000000	1·000000000	1·0000000000

Table 2

u_n	Newton–Raphson–Kantorovich $c = -0\cdot1$, $d = 1\cdot0$ $h = 0\cdot010$	$h = 0\cdot005$	$h = 0\cdot002$	$h = 0\cdot001$
$u_0(x)$	0·0 for $x\varepsilon[0, 1]$	0·0 for $x\varepsilon_0[0, 1]$	0·0 for $x\varepsilon[0, 1]$	0·0 for $x\varepsilon[0, 1]$
$u_1(0\cdot5)$	0·999404	0·99940433	0·999404329	0·9994043290
$u_2(0\cdot5)$	1·000000	1·00000000	0·999999999	0·9999999996
$u_3(0\cdot5)$	1·000000	1·00000000	1·000000000	1·0000000000

Finally, and for purposes of comparison, the modified quasilinear method discussed in Section 5 was employed to solve the same nonlinear equation. The successive approximation scheme obtained by application of this method to equation (35 and 36) is

$$\frac{d^4u_{n+1}}{dx^4} + (1 + cd\, e^{dv_n})u_{n+1} = q(x) - c\, e^{dv_n}(1 - dv_n),$$

$$u_{n+1}(0) = u''_{n+1}(0) = u_{n+1}(1) = u''_{n+1}(0) = 0, \quad (43)$$

where v_n is the solution of the original equation

$$\frac{d^4v_n}{dx^4} + v_n + c\, e^{d \sin v_n} = p(x), \quad (44)$$

but subject to the initial conditions

$$\begin{aligned} v_n(0) &= v''_n(0) = 0, \\ v'_n(0) &= u'_n(0), \\ v'''_n(0) &= u'''_n(0). \end{aligned} \quad (45)$$

The integration of the coupled system (43) and (44 and 45) was performed by reducing (43) to an initial value problem by the same method employed in the previous example. The numerical integrations were carried out by using $h = 0\cdot01$. In order to compare the results with the previous ones, we made $v_0 \equiv 0$. The results are presented in Table 3.

Table 3

$c = 5{\cdot}0,\quad d = 2{\cdot}0$					
Modified quasilinearization scheme				Newton–Raphson–Kantorovich scheme	
u_n	$h = 0{\cdot}010$	v_n	$h = 0{\cdot}010$	u_n	$h = 0{\cdot}010$
$u_0(x)$		$v_0(x)$	0·0 for $x^2[0, 1]$	$u_0(x)$	0·0 for $x^2[0, 1]$
$u_1(0{\cdot}5)$	1·156831	$v_1(0{\cdot}5)$	1·148745	$u_1(0{\cdot}5)$	1·156831
$u_2(0{\cdot}5)$	1·007273	$v_2(0{\cdot}5)$	1·006964	$u_2(0{\cdot}5)$	1·008793
$u_3(0{\cdot}5)$	1·000014	$v_3(0{\cdot}5)$	1·000014	$u_3(0{\cdot}5)$	1·000025
$u_4(0{\cdot}5)$	1·000000	$v_4(0{\cdot}5)$	1·000000	$u_4(0{\cdot}5)$	1·000000
$u_5(0{\cdot}5)$	1·000000	$v_5(0{\cdot}5)$	1·000000	$u_5(0{\cdot}5)$	1·000000

7. CONCLUDING REMARKS

In the methods previously exposed we can distinguish two separate aspects, both of which serve to highlight the power and the efficiency of quasilinearization as the underlining theory to derive successive approximation schemes for the numerical solution of nonlinear functional equations. In the first place we should emphasize the quadratic convergence of the approximating sequences previously derived. Secondly, we must mention the monotone behavior of the successive approximations. This last property depends on the positivity of Green's function of a certain associated linear operator. It should be noted that for boundary conditions other than those considered here the positivity property might not hold. This is particularly true when we consider the beam with free-ends, i.e. for which $u'' = u''' = 0$ at both ends. For these boundary conditions the positivity of the associated linear operator does not hold, even for arbitrarily small values of L. When this is the case, it is clear that the quadratic nature of the approximations remains preserved, but we cannot insure any longer, monotone approximation.

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